

Information Complexity of Quantum Gates

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This paper considers the realizability of quantum gates from the perspective of information complexity. Since the gate is a physical device that must be controlled classically, it is subject to random error. We define the complexity of gate operation in terms of the difference between the entropy of the variables associated with initial and final states of the computation. We argue that the gate operations are irreversible if there is a difference in the accuracy associated with input and output variables. It is shown that under some conditions the gate operation may be associated with unbounded entropy, implying impossibility of implementation.

KEY WORDS: quantum gates; quantum computing.

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1. INTRODUCTION

In this paper, we consider complexity and realizability of quantum gates from the point of view of information theory. A gate is a physical system that is controlled by varying some input variables, which are classical. In principle, such a physical system could implement a variety of operators based on the control variables. The gate functions may be also implemented by a single physical system that operates sequentially on the qubits in the quantum register. The complexity of the gate will be defined in terms of the entropy associated with its control. From a practical point of view, one is interested in asking how easy it is to control a gate.

As no analog system can have infinite precision, we investigate what happens if the precision levels at the input and the output are different. The complexity of the gate, defined in terms of entropy, will be examined for the rotation and CNOT gates in certain circuits.

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2. INFORMATION PROCESSING BY GATE

One aspect of gate performance is its accuracy. Researchers on quantum information science have given much attention to the question of errors and their correction (Nielsen and Chuang, 2000; Kitaev, 1997; Knill and Laflamme, 1997) by drawing upon parallels with classical information. Quantum error-correction coding works like classical error-correction to correct some large errors.

But the framework of quantum information is distinct from that of classical information. In the classical case, it is implicitly assumed that there occurs an automatic correction of errors that are smaller than a threshold by means of clipping or by the use of a decision circuit. In the case of quantum information, the input data is nominally discrete, but in reality its precision cannot be absolute in any actual realization. Furthermore, unknown small errors in quantum information cannot be corrected (Kak, 1999, 2003a). Consequently, proposals for error correction and fault tolerance (such as Steane, 1999; Svore, 2004; Knill, 2004a) remain unrealistic.

Classical analog computation and quantum processing do have parallels. In general, fixed errors in gate operation could become irreversible due to actual small nonlinearity of nominally linear elements. Analog computing is not practical to implement because noise cannot be separated from useful signal and it accumulates, degrading the system performance in an uncorrectable manner.

If there were no noise, the practicality of analog computing would depend on the feasibility of the gate implementation over the expected input-output range. This feasibility must be checked in the context of the limitations on information processing by the gate.

Consider the gate G of Fig. 1. It may be assumed that it is a physical system, which is controlled by means of some variable. This control is implemented by choosing a setting on an instrument, and this choice is associated with random error. If one views the circuit operations to be implemented by the same device transitioning through various states in sequence, then one can determine the distribution of the control variable states, and compute its entropy. This entropy, when determined for the entire computing circuit, may be taken to represent its complexity.

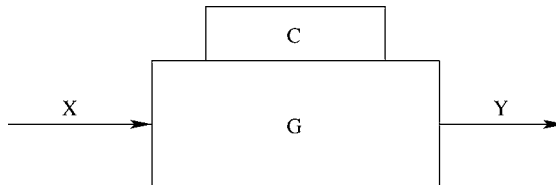


Fig. 1. Information processing gate, G , with control, C .

Information is preserved, therefore one can define the following relationship for the entropy expressions for the input X , the gate control information C , and the output Y :

$$H(Y) = H(X) + H(C). \tag{1}$$

Although it is assumed that the variable X is discrete, in reality the lack of perfect precision at the state preparation state makes it a continuous variable (Kak, 2000, 2001a,b). The lack of precision may not affect the measurement variables, but it would introduce continuous phase error.

Similarly, the output variable Y has discrete measurement associated with it, but it may come with additional component states and many unknown, continuous phase terms. This has implications for quantum amplitudes and, consequently, with the probabilities associated with the states.

As an aside, Eq. (1) provides an explanation for the no-cloning theorem. A gate cannot clone a state since this would require the gate to supply information equal to that of the unknown state, which, by virtue of its being unknown, is impossible.

As the variables X and Y (defined together with associated continuous phase terms) are continuous, the classical variable C must also be continuous. The entropy associated with a continuous variable Z is given by the expression:

$$H(Z) = h(Z) - \lim_{\Delta z \rightarrow 0} \log_2 \Delta z \tag{2}$$

where $h(Z)$ is the differential entropy:

$$h(Z) = \int_{-\infty}^{\infty} f_Z(z) \log_2 \left[\frac{1}{f_Z(z)} \right] dz \tag{3}$$

and Δz is the precision associated with the variable.

If the precision is the same at both input and output, the term $\lim_{\Delta z \rightarrow 0} \log_2 \Delta z$ will cancel out and the differential entropies would be a proper measure of the entropy of X and Y . In other words,

$$H(C) = h(Y) - h(X). \tag{4}$$

The entropy associated with $H(C)$ is the information lost in the computation process and it may be converted to heat according to thermodynamic laws (Bennett, 1982; Kak, 1998; Landauer, 1961). If $H(C)$ is nonzero, error-free quantum computation is impossible, since this is associated with loss of information.

3. MULTIPLICATION BY CONSTANT

Example 3.1. Consider a gate, which multiplies the inputs by a fixed constant $k > 1$. If the input X is distributed uniformly over the interval $(0, a)$, then the

output Y is distributed uniformly over $(0, ka)$. The differential entropy values of the input and the output are:

$$h(X) = \log_2 a \quad (5)$$

$$h(Y) = \log_2 ka \quad (6)$$

Assuming the same precision at input and output, the gate needs to supply entropy equal to $H(C) = \log_2 ka - \log_2 a = \log_2 k$, which become large as k increases. This supply of entropy will have to be done in terms of interpolation or other processing, which cannot be perfect.

If $k < 1$, then the output entropy is smaller than input entropy and, therefore, $H(C)$ represents loss of information in the output. In effect, the assumption of fixed amplification of a variable with the same absolute precision at the output amounts to a nonlinear, irreversible process. For example, when a picture is compressed, one cannot obtain the original to the earlier precision by amplifying it back. In practical terms, the precision needed for the realization of a universal gate will be unattainable for a variety of reasons: one cannot have perfectly linear behavior in an electrical circuit over an unrestricted range. Unrestricted multiplication of a continuous variable is not implementable if the precision remains unchanged.

In quantum computing, problems that somewhat parallel this above example are the implementation of rotation and CNOT gates, two operators that are basic to the computation process (DiVincenzo, 1995). The necessarily classical control of the gate is marred by random errors as well as calibration errors.

4. ROTATION

Example 4.2 Consider a quantum gate that rotates the input qubit by a fixed angle. Since the input X and the output Y will be distributed uniformly over the same interval $(0, a)$, the entropy associated with this gate will be 0 (as per Eq. (4)) as is required by the reversible nature of the assumed quantum evolution.

But if the precision associated with the measurement and initialization processes at the input and the output is different, then lossless (or, equivalently, error-free) evolution cannot be assumed.

5. CNOT AND HADAMARD GATES

Consider the CNOT gate together with a companion Hadamard gate. The errors in the device implementation of the CNOT gate may make the gate effectively nonlinear and hence nonunitary. The matrix values that the device embodies may

be different from the nominal ones below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

For simplicity, we consider a very straightforward situation, which does not affect the CNOT gate, but where its companion Hadamard gate is off the correct value, stuck in the state

$$H_s = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \tag{8}$$

where $\theta \neq 45^\circ$.

6. STUCK HADAMARD GATE BEFORE A CNOT

Example 6.3. Consider the arrangement of Fig. 2, where the stuck gate H_s ($\theta \neq \pi/4$, but its value is known) is to the left of the CNOT gate; this circuit demonstrates that quantum processing can compute a global property of a function by a single measurement (Nielsen and Chuang, 2000).

It will be seen that at the output of the CNOT gate, the state is:

$$\frac{1}{\sqrt{2}} (\cos \theta |0\rangle - \sin \theta |1\rangle) (|0\rangle - |1\rangle)$$

The state $|a\rangle = (\cos \theta |0\rangle - \sin \theta |1\rangle)$, which is in error, may be passed through the gate $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ followed by another H_s to yield $|0\rangle$, which can be transformed to the correct $|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. In this example, the state $|b\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ was not affected by the stuck gate H_s .

When the stuck gate is the lower Hadamard gate, as in Fig. 3, the state at the output of the CNOT gate is:

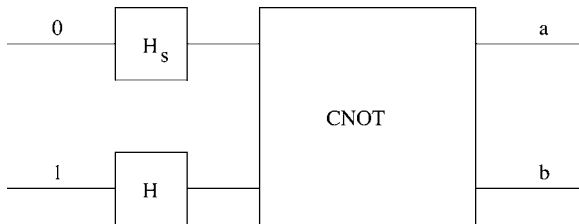


Fig. 2. The stuck Hadamard gate H_s before CNOT.

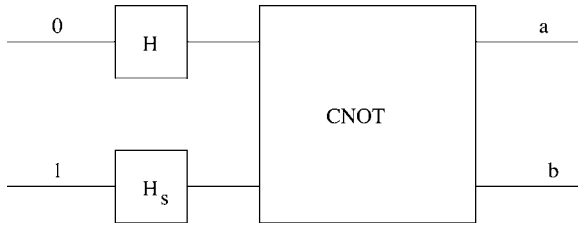


Fig. 3. The stuck Hadamard gate H_s before CNOT in the lower input.

$$\frac{1}{\sqrt{2}}(\sin \theta|00\rangle - \cos \theta|01\rangle + \sin \theta|11\rangle - \cos \theta|10\rangle)$$

Corresponding to this we have the density function ρ^{ab} given below:

$$\rho^{ab} = \frac{1}{2} \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta & -\sin \theta \cos \theta & \sin^2 \theta \\ -\sin \theta \cos \theta & \cos^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \cos^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta \\ \sin^2 \theta & -\sin \theta \cos \theta & -\sin \theta \cos \theta & \sin^2 \theta \end{bmatrix}$$

It follows that the reduced density matrix for the state $|a\rangle$ is:

$$\rho^a = \frac{1}{2} \begin{bmatrix} 1 & -\sin 2\theta \\ -\sin 2\theta & 1 \end{bmatrix}$$

Therefore, when $\theta \neq \pi/4$, ρ^{ab} is a mixture, and we cannot perform any local correction to $|a\rangle$ to obtain the correct product state, for a unitary transformation on a mixture will keep it as a mixture. In other words, this error is not locally correctable.

7. STUCK HADAMARD GATE IN THE TELEPORTATION PROTOCOL

Example 7.4. In the teleportation protocol, an unknown quantum state (of particle X) is teleported to a remote location using two entangled particles (Y and Z) and classical information. Here, for convenience, we use the variant teleportation protocol (Kak, 2003b), which requires only one classical bit in its classical information link (Fig. 4). But instead of the Hadamard operator, we consider H_s to be the rotation operator with angle θ . We assume that the receiver has a copy of H_s available for local processing, and we would like to estimate what would happen if this copy is not identical to the one used at the transmitting end.

The state X is $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are unknown amplitudes, and Y and Z are in the pure entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The initial state of

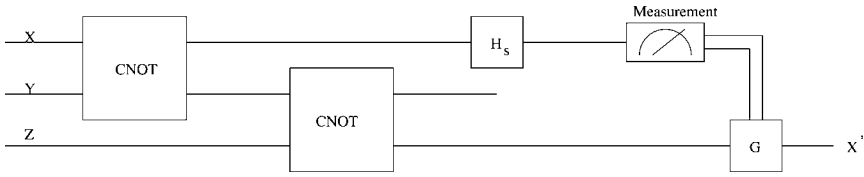


Fig. 4. The stuck Hadamard gate H_s in the teleportation set-up.

the three particles is:

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \beta|100\rangle + \alpha|011\rangle + \beta|111\rangle)$$

The sequence of steps in Fig. 4 is as follows:

1. Apply chained transformations: CNOT on X and Y , followed by CNOT on Y and Z .
2. Apply H_s on the state of X .
3. Measure the state of X and transfer information regarding it.
4. Apply appropriate operator G to complete teleportation of the unknown state.

A simple calculation will show that the state before the measurement is:

$$\begin{aligned} &\frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle)(\alpha \cos \theta|0\rangle + \beta \sin \theta|1\rangle) \\ &+ \frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle)(\alpha \sin \theta|0\rangle - \beta \cos \theta|1\rangle) \end{aligned} \tag{9}$$

Therefore, after the measurement, we get either

$$X^+ = \alpha \cos \theta|0\rangle + \beta \sin \theta|1\rangle$$

or

$$X^- = \alpha \sin \theta|0\rangle - \beta \cos \theta|1\rangle$$

based on whether the measurement was 0 or 1. Assuming that the value of θ is also communicated to it, the receiver can recover the unknown X probabilistically; when the value of θ is 45° , then the inversion is trivially simple.

For simplicity, assume that the receiver needs to invert X^+ . He will replicate Fig. 4 at his end, which means that the Hadamard gate that he would use would have identical characteristics (the same precision) to the one used during the earlier operation. He would now obtain either

$$X^{++} = \alpha \cos^2 \theta|0\rangle + \beta \sin^2 \theta|1\rangle$$

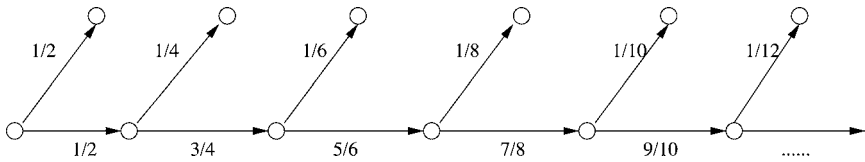


Fig. 5. The probability tree for recovering the state X.

or

$$X^{+-} = \alpha|0\rangle + \beta|1\rangle$$

Similarly, X^- will, in the next iteration, lead to:

$$X^{-+} = \alpha|0\rangle + \beta|1\rangle$$

or

$$X^{--} = \alpha \sin^2 \theta |0\rangle + \beta \cos^2 \theta |1\rangle$$

This procedure may be extended, and the probability of recovering the unknown state X can be shown to be given by a tree diagram of Fig. 5.

In the first pass, there is a $\frac{1}{2}$ probability of getting the correct state, and this probability reduces in further passes to $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$, and so on. The probability of recovering the state X is thus:

$$\frac{1}{2} + \frac{1}{2} \frac{1}{4} + \frac{1}{2} \frac{1}{4} \frac{1}{6} + \frac{1}{2} \frac{1}{4} \frac{1}{6} \frac{1}{8} + \frac{1}{2} \frac{1}{4} \frac{1}{6} \frac{1}{8} \frac{1}{10} + \dots \tag{10}$$

The ability of the receiver to implement the needed transformation will depend on the precision available in its gate control mechanism. If the value of θ at the sending point is smaller than the precision available to the receiver, then the state X cannot be recovered.

It is interesting that as long as the receiver possesses a rotation operator H_s that is identical to the one used at the sending point, there is no need to know the value of θ and still obtain the unknown state X probabilistically, as in expression (9).

8. CONCLUSION

We have considered the problem of gate complexity in quantum systems. The control of the gate—a physical device—is by modifying some classical variable, which is subject to error. Since one cannot assume infinite precision in any control system, the implications of varying accuracy amongst different gates becomes an important problem.

We have shown that in certain arrangements a stuck fault cannot be reversed down the circuit stream using a single qubit operator, for it converted a pure state into a mixed state.

We considered the case of the teleportation circuit with the rotation gate stuck at θ . When $\theta = 0^\circ$, the state X collapses to 0 or 1. When $\theta \neq 0^\circ$ or 90° , one may obtain the unknown state back probabilistically by passing X^+ or X^- back through the circuit of Fig. 4 iteratively.

Consider two parties, A and B , who are both presented with the state X^+ or X^- . If the precision available to one of them is greater than or equal to that of the sender, and that of the other is less, then one of them can recover the state, whereas the other cannot.

It is essential that the entropy rate associated with the quantum circuit be smaller than what can be implemented by the information capacity of the controller. This perspective may be useful in evaluating proposals (Knill, 2004b) for quantum computing with noisy components.

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